

# THE $n$ LINEAR EMBEDDING THEOREM

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ABSTRACT. Let  $\sigma_i$ ,  $i = 1, \dots, n$ , denote positive Borel measures on  $\mathbb{R}^d$ , let  $\mathcal{D}$  denote the usual collection of dyadic cubes in  $\mathbb{R}^d$  and let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map. In this paper we give a characterization of the  $n$  linear embedding theorem. That is, we give a characterization of the inequality

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_i d\sigma_i \right| \leq C \prod_{i=1}^n \|f_i\|_{L^{p_i}(d\sigma_i)}$$

in terms of multilinear Sawyer's checking condition and discrete multilinear Wolff's potential, when  $1 < p_i < \infty$ .

## 1. INTRODUCTION

The purpose of this paper is to investigate the  $n$  linear embedding theorem. We first fix some notations. We will denote by  $\mathcal{D}$  the family of all dyadic cubes  $Q = 2^{-k}(m + [0, 1)^d)$ ,  $k \in \mathbb{Z}$ ,  $m \in \mathbb{Z}^d$ . Let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map and let  $\sigma_i$ ,  $i = 1, \dots, n$ , be positive Borel measures on  $\mathbb{R}^d$ . In this paper we give a necessary and sufficient condition for which the inequality

$$(1.1) \quad \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_i d\sigma_i \right| \leq C \prod_{i=1}^n \|f_i\|_{L^{p_i}(d\sigma_i)},$$

to hold when  $1 < p_i < \infty$ .

For the bilinear embedding theorem, in the case  $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$ , Sergei Treil gives a simple proof of the following.

**Proposition 1.1** ([9, Theorem 2.1]). *Let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map and let  $\sigma_i$ ,  $i = 1, 2$ , be positive Borel measures on  $\mathbb{R}^d$ . Let  $1 < p_i < \infty$  and  $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$ . The following statements are equivalent:*

(a) *The following bilinear embedding theorem holds:*

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^2 \left| \int_Q f_i d\sigma_i \right| \leq c_1 \prod_{i=1}^2 \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;$$

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(b) For all  $Q \in \mathcal{D}$ ,

$$\begin{cases} \left( \int_Q \left( \sum_{Q' \subset Q} K(Q') \sigma_1(Q') 1_{Q'} \right)^{p'_2} d\sigma_2 \right)^{1/p'_2} \leq c_2 \sigma_1(Q)^{1/p_1} < \infty, \\ \left( \int_Q \left( \sum_{Q' \subset Q} K(Q') \sigma_2(Q') 1_{Q'} \right)^{p'_1} d\sigma_1 \right)^{1/p'_1} \leq c_2 \sigma_2(Q)^{1/p_2} < \infty. \end{cases}$$

Moreover, the least possible  $c_1$  and  $c_2$  are equivalent.

Here, for each  $1 < p < \infty$ ,  $p'$  denote the dual exponent of  $p$ , i.e.,  $p' = \frac{p}{p-1}$ , and  $1_E$  stands for the characteristic function of the set  $E$ .

Proposition 1.1 was first proved for  $p_1 = p_2 = 2$  in [4] by the Bellman function method. Later in [3], this was proved in full generality. The checking condition in Proposition 1.1 is called “the Sawyer type checking condition”, since this was first introduced by Eric T. Sawyer in [5, 6].

To describe the case  $\frac{1}{p_1} + \frac{1}{p_2} < 1$ , we need discrete Wolff’s potential.

Let  $\mu$  and  $\nu$  be positive Borel measures on  $\mathbb{R}^d$  and let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map. For  $p > 1$ , the discrete Wolff’s potential  $\mathcal{W}_{K,\mu}^p[\nu](x)$  of the measure  $\nu$  is defined by

$$\mathcal{W}_{K,\mu}^p[\nu](x) := \sum_{Q \in \mathcal{D}} K(Q) \mu(Q) \left( \frac{1}{\mu(Q)} \sum_{Q' \subset Q} K(Q') \mu(Q') \nu(Q') \right)^{p-1} 1_Q(x), \quad x \in \mathbb{R}^d.$$

The author prove the following.

**Proposition 1.2** ([7, Theorem 1.3]). *Let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map and let  $\sigma_i$ ,  $i = 1, 2$ , be positive Borel measures on  $\mathbb{R}^d$ . Let  $1 < p_i < \infty$  and  $\frac{1}{p_1} + \frac{1}{p_2} < 1$ . The following statements are equivalent:*

(a) *The following bilinear embedding theorem holds:*

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^2 \left| \int_Q f_i d\sigma_i \right| \leq c_1 \prod_{i=1}^2 \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;$$

(b) *For  $\frac{1}{r} + \frac{1}{p_1} + \frac{1}{p_2} = 1$ ,*

$$\begin{cases} \|\mathcal{W}_{K,\sigma_2}^{p'_2}[\sigma_1]^{1/p'_2}\|_{L^r(d\sigma_1)} \leq c_2 < \infty, \\ \|\mathcal{W}_{K,\sigma_1}^{p'_1}[\sigma_2]^{1/p'_1}\|_{L^r(d\sigma_2)} \leq c_2 < \infty. \end{cases}$$

Moreover, the least possible  $c_1$  and  $c_2$  are equivalent.

In his exccerent survey of the  $A_2$  theorem [2], Tuomas P. Hytönen introduces another proof of Proposition 1.1, which uses the “parallel corona” decomposition. In this paper, following Hytönen’s arguments in [2], we shall establish the following theorems (Theorems 1.3 and 1.4).

**Theorem 1.3.** *Let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map and let  $\sigma_i$ ,  $i = 1, \dots, n$ , be positive Borel measures on  $\mathbb{R}^d$ . Let  $1 < p_i < \infty$  and  $\sum_{i=1}^n \frac{1}{p_i} \geq 1$ . The following statements are equivalent:*

(a) *The following  $n$  linear embedding theorem holds:*

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_i d\sigma_i \right| \leq c_1 \prod_{i=1}^n \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;$$

(b) *For all  $j = 1, \dots, n$  and for all  $Q \in \mathcal{D}$ ,*

$$\sum_{Q' \subset Q} K(Q') \sigma_j(Q') \prod_{\substack{i=1 \\ i \neq j}}^n \left| \int_{Q'} f_i d\sigma_i \right| \leq c_2 \sigma_j(Q)^{1/p_j} \prod_{\substack{i=1 \\ i \neq j}}^n \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty.$$

Moreover, the least possible  $c_1$  and  $c_2$  are equivalent.

Let the symmetric group  $S_n$  be the set of all permutations of the set  $\{1, \dots, n\}$ , that is, the set of all bijections from the set  $\{1, \dots, n\}$  to itself. Let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map and let  $\sigma_i$ ,  $i = 1, \dots, n$ , be positive Borel measures on  $\mathbb{R}^d$ . Let  $1 < p_i < \infty$  and  $\sum_{i=1}^n \frac{1}{p_i} < 1$ .

Let  $\phi \in S_n$ . Set

$$\begin{aligned} \frac{1}{r_1^\phi} + \frac{1}{p_{\phi(1)}} &= 1, \\ \frac{1}{r_2^\phi} + \frac{1}{p_{\phi(1)}} + \frac{1}{p_{\phi(2)}} &= 1, \\ &\vdots \\ \frac{1}{r_{n-1}^\phi} + \sum_{i=1}^{n-1} \frac{1}{p_{\phi(i)}} &= 1, \\ \frac{1}{r} + \sum_{i=1}^n \frac{1}{p_{\phi(i)}} &= 1. \end{aligned}$$

Let, for  $Q \in \mathcal{D}$ ,

$$K_1^\phi(Q) := K(Q) \sigma_{\phi(1)}(Q) \left( \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \prod_{i=1}^n \sigma_{\phi(i)}(Q') \right)^{r_1^\phi - 1},$$

let

$$K_2^\phi(Q) := K_1^\phi(Q) \sigma_{\phi(2)}(Q) \left( \frac{1}{\sigma_{\phi(2)}(Q)} \sum_{Q' \subset Q} K_1^\phi(Q') \prod_{i=2}^n \sigma_{\phi(i)}(Q') \right)^{r_2^\phi / r_1^\phi - 1}$$

and, inductively, for  $j = 3, \dots, n-1$ , let

$$K_j^\phi(Q) := K_{j-1}^\phi(Q) \sigma_{\phi(j)}(Q) \left( \frac{1}{\sigma_{\phi(j)}(Q)} \sum_{Q' \subset Q} K_{j-1}^\phi(Q') \prod_{i=j}^n \sigma_{\phi(i)}(Q') \right)^{r_j^\phi / r_{j-1}^\phi - 1}.$$

**Theorem 1.4.** *With the notation above, the following statements are equivalent:*

(a) *The following  $n$  linear embedding theorem holds:*

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_i d\sigma_i \right| \leq c_1 \prod_{i=1}^n \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;$$

(b) For all  $\phi \in S_n$ ,

$$\left\| \left( \sum_{Q \in \mathcal{D}} K_{n-1}^\phi(Q) 1_Q \right)^{1/r_{n-1}^\phi} \right\|_{L^r(d\sigma_{\phi(n)})} \leq c_2 < \infty.$$

Moreover, the least possible  $c_1$  and  $c_2$  are equivalent.

Even though Theorems 1.3 and 1.4 both characterize the same  $n$  linear embedding theorem, it seems that the characterizations are very different. In very recent paper [1], Timo S. Hänninen, Tuomas P. Hytönen and Kangwei Li give a unified approach saying “sequential testing” characterization, when  $n = 2, 3$ . Especially, our Theorem 1.4 with  $n = 3$  is obtained in [1, Theorem 1.16]. (An alternative form of another unified characterization has been simultaneously obtained by Vuorinen [10].) In [8], the author gives a characterization of the trilinear embedding theorem in terms of Theorem 1.3 and Propositions 1.1 and 1.2.

The letter  $C$  will be used for constants that may change from one occurrence to another.

## 2. PROOF OF THE NECESSITY

In what follows we shall prove the necessity of theorems. The necessity of Theorem 1.3, that is, (b) follows from (a) at once if we substitute the test function  $f_j = 1_Q$ . So, we shall verify the necessity of Theorem 1.4. We need a lemma (cf. Lemma 2.1 in [7]).

**Lemma 2.1.** *Let  $\sigma$  be a positive Borel measure on  $\mathbb{R}^d$ . Let  $1 < s < \infty$  and  $\{\alpha_Q\}_{Q \in \mathcal{D}} \subset [0, \infty)$ . Define, for  $Q_0 \in \mathcal{D}$ ,*

$$\begin{aligned} A_1 &:= \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^s d\sigma, \\ A_2 &:= \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1}, \\ A_3 &:= \int_{Q_0} \sup_{Q \subset Q_0} \left( \frac{1_Q(x)}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^s d\sigma(x). \end{aligned}$$

Then

$$A_1 \leq c(s)A_2, \quad A_2 \leq c(s)^{\frac{1}{s-1}}A_3 \quad \text{and} \quad A_3 \leq (s')^s A_1.$$

Here,

$$c(s) := \begin{cases} s, & 1 < s \leq 2, \\ (s(s-1) \cdots (s-k))^{\frac{s-1}{s-k-1}}, & 2 < s < \infty, \end{cases}$$

where  $k = \lceil s - 2 \rceil$  is the smallest integer greater than  $s - 2$ .

We will use  $f_Q f d\sigma$  to denote the integral average  $\sigma(Q)^{-1} \int_Q f d\sigma$ . The dyadic maximal operator  $M_{\mathcal{D}}^\sigma$  is defined by

$$M_{\mathcal{D}}^\sigma f(x) := \sup_{Q \in \mathcal{D}} \frac{1_Q(x)}{\sigma(Q)} \int_Q |f(y)| d\sigma(y).$$

Suppose that (a) of Theorem 1.4. Then, for  $\phi \in S_n$ ,

$$(2.1) \quad \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right| \leq c_1 \prod_{i=1}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.$$

Recall that  $\frac{1}{r_1^\phi} + \frac{1}{p_{\phi(1)}} = 1$ . By duality, we see that

$$\int_{\mathbb{R}^d} \left( \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=2}^n \left| \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right| 1_Q \right)^{r_1^\phi} d\sigma_{\phi(1)} \leq c_1^{r_1^\phi} \prod_{i=2}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{r_1^\phi},$$

which implies by Lemma 2.1

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} K(Q) \sigma_{\phi(1)}(Q) \prod_{i=2}^n \left| \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right| \\ & \quad \times \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \sigma_{\phi(1)}(Q') \prod_{i=2}^n \left| \int_{Q'} f_{\phi(i)} d\sigma_{\phi(i)} \right| \right]^{r_1^\phi - 1} \\ & \leq C c_1^{r_1^\phi} \prod_{i=2}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{r_1^\phi}. \end{aligned}$$

It follows from this inequality that

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} K_1^\phi(Q) \prod_{i=2}^n \left| \int_Q g_{\phi(i)} d\sigma_{\phi(i)} \right| \\ & = \sum_{Q \in \mathcal{D}} K(Q) \sigma_{\phi(1)}(Q) \prod_{i=2}^n \left| \int_Q g_{\phi(i)} d\sigma_{\phi(i)} \right| \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \prod_{i=1}^n \sigma_{\phi(i)}(Q') \right]^{r_1^\phi - 1} \\ & = \sum_{Q \in \mathcal{D}} K(Q) \sigma_{\phi(1)}(Q) \prod_{i=2}^n \sigma_{\phi(i)}(Q) \left| \int_Q g_{\phi(i)} d\sigma_{\phi(i)} \right|^{1/r_1^\phi} \\ & \quad \times \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \sigma_{\phi(1)}(Q') \prod_{i=2}^n \sigma_{\phi(i)}(Q') \left| \int_{Q'} g_{\phi(i)} d\sigma_{\phi(i)} \right|^{1/r_1^\phi} \right]^{r_1^\phi - 1} \\ & \leq \sum_{Q \in \mathcal{D}} K(Q) \sigma_{\phi(1)}(Q) \prod_{i=2}^n \int_Q (M_{\mathcal{D}}^{\sigma_{\phi(i)}} g_{\phi(i)})^{1/r_1^\phi} d\sigma_{\phi(i)} \\ & \quad \times \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \sigma_{\phi(1)}(Q') \prod_{i=2}^n \int_{Q'} (M_{\mathcal{D}}^{\sigma_{\phi(i)}} g_{\phi(i)})^{1/r_1^\phi} d\sigma_{\phi(i)} \right]^{r_1^\phi - 1} \\ & \leq C c_1^{r_1^\phi} \prod_{i=2}^n \|M_{\mathcal{D}}^{\sigma_{\phi(i)}} g_{\phi(i)}\|_{L^{p_{\phi(i)}/r_1^\phi}(d\sigma_{\phi(i)})} \\ & \leq C c_1^{r_1^\phi} \prod_{i=2}^n \|g_{\phi(i)}\|_{L^{p_{\phi(i)}/r_1^\phi}(d\sigma_{\phi(i)})}, \end{aligned}$$

where we have used the boundedness of dyadic maximal operators. Thus, we obtain

$$(2.2) \quad \sum_{Q \in \mathcal{D}} K_1^\phi(Q) \prod_{i=2}^n \left| \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right| \leq C c_1^{r_1^\phi} \prod_{i=2}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}/r_1^\phi}(d\sigma_{\phi(i)})}.$$

Notice that

$$(2.3) \quad \begin{cases} \frac{r_{i-1}^\phi}{r_i^\phi} + \frac{r_{i-1}^\phi}{p_{\phi(i)}} = 1, & i = 2, \dots, n-1, \\ \frac{r_{n-1}^\phi}{r} + \frac{r_{n-1}^\phi}{p_{\phi(n)}} = 1. \end{cases}$$

By the same manner as the above but starting from (2.2), instead of (2.1), and using (2.3) with  $i = 2$ , we obtain

$$\sum_{Q \in \mathcal{D}} K_2^\phi(Q) \prod_{i=3}^n \left| \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right| \leq C c_1^{r_2^\phi} \prod_{i=3}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}/r_2^\phi}(d\sigma_{\phi(i)})}.$$

By being continued inductively until the  $n-1$  step, we obtain

$$\sum_{Q \in \mathcal{D}} K_{n-1}^\phi(Q) \left| \int_Q f_{\phi(n)} d\sigma_{\phi(n)} \right| \leq C c_1^{r_{n-1}^\phi} \|f_{\phi(n)}\|_{L^{p_{\phi(n)}/r_{n-1}^\phi}(d\sigma_{\phi(n)})}.$$

Notice that the last equation of (2.3). Then by duality

$$\left\| \sum_{Q \in \mathcal{D}} K_{n-1}^\phi(Q) 1_Q \right\|_{L^{r/r_{n-1}^\phi}(d\sigma_{\phi(n)})} \leq C c_1^{r_{n-1}^\phi}$$

and, hence,

$$\left\| \left( \sum_{Q \in \mathcal{D}} K_{n-1}^\phi(Q) 1_Q \right)^{1/r_{n-1}^\phi} \right\|_{L^r(d\sigma_{\phi(n)})} \leq C c_1,$$

which completes the necessity of Theorem 1.4.

### 3. PROOF OF THE SUFFICIENCY

In what follows we shall prove the sufficiency of theorems.

Let  $Q_0 \in \mathcal{D}$  be taken large enough and be fixed. We shall estimate the quantity

$$(3.1) \quad \sum_{Q \subset Q_0} K(Q) \prod_{i=1}^n \left( \int_Q f_i d\sigma_i \right),$$

where  $f_i \in L^{p_i}(d\sigma_i)$  is nonnegative and is supported in  $Q_0$ . We define the collection of principal cubes  $\mathcal{F}_i$  for the pair  $(f_i, \sigma_i)$ ,  $i = 1, \dots, n$ . Namely,

$$\mathcal{F}_i := \bigcup_{k=0}^{\infty} \mathcal{F}_i^k,$$

where  $\mathcal{F}_i^0 := \{Q_0\}$ ,

$$\mathcal{F}_i^{k+1} := \bigcup_{F \in \mathcal{F}_i^k} ch_{\mathcal{F}_i}(F)$$

and  $ch_{\mathcal{F}_i}(F)$  is defined by the set of all “maximal” dyadic cubes  $Q \subset F$  such that

$$\int_Q f_i d\sigma_i > 2 \int_F f_i d\sigma_i.$$

Observe that

$$\begin{aligned}
& \sum_{F' \in ch_{\mathcal{F}_i}(F)} \sigma_i(F') \\
& \leq \left( 2 \int_F f_i d\sigma_i \right)^{-1} \sum_{F' \in ch_{\mathcal{F}_i}(F)} \int_{F'} f_i d\sigma_i \\
& \leq \left( 2 \int_F f_i d\sigma_i \right)^{-1} \int_F f_i d\sigma_i = \frac{\sigma_i(F)}{2},
\end{aligned}$$

which implies

$$(3.2) \quad \sigma_i(E_{\mathcal{F}_i}(F)) := \sigma_i \left( F \setminus \bigcup_{F' \in ch_{\mathcal{F}_i}(F)} F' \right) \geq \frac{\sigma_i(F)}{2},$$

where the sets  $E_{\mathcal{F}_i}(F)$ ,  $F \in \mathcal{F}_i$ , are pairwise disjoint. We further define the stopping parents, for  $Q \in \mathcal{D}$ ,

$$\begin{cases} \pi_{\mathcal{F}_i}(Q) := \min\{F \supset Q : F \in \mathcal{F}_i\}, \\ \pi(Q) := (\pi_{\mathcal{F}_1}(Q), \dots, \pi_{\mathcal{F}_n}(Q)). \end{cases}$$

Then we can rewrite the series in (3.1) as follows:

$$\sum_{Q \subset Q_0} = \sum_{(F_1, \dots, F_n) \in (\mathcal{F}_1, \dots, \mathcal{F}_n)} \sum_{\substack{Q: \\ \pi(Q) = (F_1, \dots, F_n)}}.$$

We notice the elementary fact that, if  $P, R \in \mathcal{D}$ , then  $P \cap R \in \{P, R, \emptyset\}$ . This fact implies, if  $\pi(Q) = (F_1, \dots, F_n)$ , then

$$Q \subset F_{\phi(1)} \subset \dots \subset F_{\phi(n)} \quad \text{for some } \phi \in S_n.$$

Thus, for fixed  $\phi \in S_n$ , we shall estimate

$$(3.3) \quad \sum_{\substack{(F_{\phi(i)}) \in (\mathcal{F}_{\phi(i)}): \\ F_{\phi(1)} \subset \dots \subset F_{\phi(n)}}} \sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)}}} K(Q) \prod_{i=1}^n \left( \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right).$$

**Proof of (a) of Theorem 1.3.** It follows that, for fixed  $F_{\phi(n)} \in \mathcal{F}_{\phi(n)}$ ,

$$\begin{aligned}
& \sum_{F_{\phi(1)} \subset \dots \subset F_{\phi(n)}} \sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)}}} K(Q) \prod_{i=1}^n \left( \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right) \\
& \leq \left( 2 \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right) \sum_{F_{\phi(1)} \subset \dots \subset F_{\phi(n)}} \sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)}}} K(Q) \sigma_{\phi(n)}(Q) \prod_{i=1}^{n-1} \left( \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right).
\end{aligned}$$

We need two observations. Suppose that  $F_{\phi(1)} \subset \dots \subset F_{\phi(n)}$  and  $\pi(Q) = (F_{\phi(i)})$ . Let  $i = 1, \dots, n-1$ . If  $F' \in ch_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})$  satisfies  $F' \subset Q$ . Then

$$(3.4) \quad \pi_{\mathcal{F}_{\phi(n)}}(\pi_{\mathcal{F}_{\phi(i)}}(F')) = \begin{cases} F_{\phi(n)}, & \text{when } f' \notin \mathcal{F}_{\phi(i)}, \\ F', & \text{when } f' \in \mathcal{F}_{\phi(i)}. \end{cases}$$

By this observation, we define

$$ch_{\mathcal{F}_{\phi(n)}}^{\phi(i)}(F_{\phi(n)}) := \{F' \in ch_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}) : F' \text{ satisfies (3.4)}\}.$$

We further observe that, when  $F' \in ch_{\mathcal{F}_{\phi(n)}}^{\phi(i)}(F_{\phi(n)})$ , we can regard  $f_{\phi(i)}$  as a constant on  $F'$  in the above integrals, that is, we can replace  $f_{\phi(i)}$  by  $f_{\phi(i)}^{F_{\phi(n)}}$  in the above integrals, where

$$f_{\phi(i)}^{F_{\phi(n)}} := f_{\phi(i)} 1_{E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})} + \sum_{F' \in ch_{\mathcal{F}_{\phi(n)}}^{\phi(i)}(F_{\phi(n)})} \left( \int_{F'} f_{\phi(i)} d\sigma_{\phi(i)} \right) 1_{F'}.$$

It follows from (b) of Theorem 1.3 that

$$\begin{aligned} & \sum_{F_{\phi(1)} \subset \cdots \subset F_{\phi(n)}} \sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)})}} K(Q) \sigma_{\phi(n)}(Q) \prod_{i=1}^{n-1} \left( \int_Q f_{\phi(i)}^{F_{\phi(n)}} d\sigma_{\phi(i)} \right) \\ & \leq c_2 \sigma_{\phi(n)}(F_{\phi(n)})^{1/p_{\phi(n)}} \prod_{i=1}^{n-1} \|f_{\phi(i)}^{F_{\phi(n)}}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}. \end{aligned}$$

Thus, we obtain

$$(3.3) \leq C c_2 \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \prod_{i=1}^{n-1} \|f_{\phi(i)}^{F_{\phi(n)}}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right) \sigma_{\phi(n)}(F_{\phi(n)})^{1/p_{\phi(n)}}.$$

Since  $\sum_{i=1}^n \frac{1}{p_{\phi(i)}} \geq 1$ , we can select the auxiliary parameters  $s_{\phi(i)}$ ,  $i = 1, \dots, n-1$ , that satisfy

$$\sum_{i=1}^{n-1} \frac{1}{s_{\phi(i)}} + \frac{1}{p_{\phi(n)}} = 1 \quad \text{and} \quad 1 < p_{\phi(i)} \leq s_{\phi(i)} < \infty.$$

It follows from Hölder's inequality with exponents  $s_{\phi(1)}, \dots, s_{\phi(n-1)}, p_{\phi(n)}$  that

$$\begin{aligned} (3.3) & \leq C c_2 \prod_{i=1}^{n-1} \left[ \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \|f_{\phi(i)}^{F_{\phi(n)}}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{s_{\phi(i)}} \right]^{1/s_{\phi(i)}} \\ & \quad \times \left[ \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{p_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/p_{\phi(n)}} \\ & \leq C c_2 \prod_{i=1}^{n-1} \left[ \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \|f_{\phi(i)}^{F_{\phi(n)}}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{p_{\phi(i)}} \right]^{1/p_{\phi(i)}} \\ & \quad \times \left[ \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{p_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/p_{\phi(n)}} \\ & =: C c_2 (I_1) \times \cdots \times (I_n), \end{aligned}$$

where we have used  $\|\cdot\|_{L^{p_{\phi(i)}}} \geq \|\cdot\|_{L^{s_{\phi(i)}}}$ .

For  $(I_n)$ , using  $\sigma_{\phi(n)}(F_{\phi(n)}) \leq 2\sigma_{\phi(n)}(E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}))$  (see (3.2)), the fact that

$$\int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \leq \inf_{y \in F_{\phi(n)}} M_{\mathcal{D}}^{\sigma_{\phi(n)}} f_{\phi(n)}(y)$$



and the disjointness of the sets  $E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})$ , we have

$$\begin{aligned} (I_n) &\leq C \left[ \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \int_{E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})} (M_{\mathcal{D}}^{\sigma_{\phi(n)}} f_{\phi(n)})^{p_{\phi(n)}} d\sigma_{\phi(n)} \right]^{1/p_{\phi(n)}} \\ &\leq C \left[ \int_{Q_0} (M_{\mathcal{D}}^{\sigma_{\phi(n)}} f_{\phi(n)})^{p_{\phi(n)}} d\sigma_{\phi(n)} \right]^{1/p_{\phi(n)}} \leq C \|f_{\phi(n)}\|_{L^{p_{\phi(n)}}(d\sigma_{\phi(n)})}. \end{aligned}$$

It remains to estimate  $(I_i)$ ,  $i = 1, \dots, n-1$ . It follows that

$$\begin{aligned} (I_i)^{p_{\phi(i)}} &= \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \int_{E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \\ &\quad + \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \sum_{F' \in \text{ch}_{\mathcal{F}_{\phi(n)}}^{\phi(i)}(F_{\phi(n)})} \left( \int_{F'} f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F'). \end{aligned}$$

By the pairwise disjointness of the sets  $E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})$ , it is immediate that

$$\sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \int_{E_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)})} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \leq \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{p_{\phi(i)}}.$$

For the remaining double sum, there holds by the uniqueness of the parent

$$\begin{aligned} &\sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \sum_{\substack{F' \in \text{ch}_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}): \\ F' \text{ satisfies (3.4)}}} \left( \int_{F'} f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F') \\ &\leq 2 \sum_{F_{\phi(n)} \in \mathcal{F}_{\phi(n)}} \sum_{\substack{F \in \mathcal{F}_{\phi(i)}: \\ \pi_{\mathcal{F}_{\phi(n)}}(F) = F_{\phi(n)}}} \sum_{\substack{F' \in \text{ch}_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}): \\ \pi_{\mathcal{F}_{\phi(i)}}(F') = F}} \left( \int_{F'} f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F') \\ &\leq 2 \sum_{F \in \mathcal{F}_{\phi(i)}} \left( 2 \int_F f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F) \\ &\leq C \|M_{\mathcal{D}}^{\sigma_{\phi(i)}} f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{p_{\phi(i)}} \leq C \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{p_{\phi(i)}}. \end{aligned}$$

Altogether, we obtain

$$(3.3) \leq C c_2 \prod_{i=1}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.$$

This yields (a) of Theorem 1.3.

**Proof of (a) of Theorem 1.4.** We shall estimate (3.3) by use of multilinear Wolff's potential. We first observe that if  $F_{\phi(i)} \in \mathcal{F}_{\phi(i)}$ ,  $i = 1, \dots, n$ , satisfy  $F_{\phi(1)} \subset \dots \subset F_{\phi(n)}$  and, for some  $Q \in \mathcal{D}$ ,  $\pi(Q) = (F_{\phi(i)})$ , then

$$(3.5) \quad \pi_{\mathcal{F}_{\phi(j)}}(F_{\phi(i)}) = F_{\phi(j)} \quad \text{for all } 1 \leq i < j \leq n.$$

Fix  $F_{\phi(i)} \in \mathcal{F}_{\phi(i)}$ ,  $i = 1, \dots, n$ , that satisfy (3.5). Then

$$\begin{aligned} &\sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)})}} K(Q) \prod_{i=1}^n \left( \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right) \\ &\leq \prod_{i=1}^n \left( 2 \int_{F_{\phi(i)}} f_{\phi(i)} d\sigma_{\phi(i)} \right) \sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)})}} K(Q) \prod_{i=1}^n \sigma_{\phi(i)}(Q). \end{aligned}$$

Recall that

$$(3.6) \quad \begin{cases} \frac{1}{r_j^\phi} + \sum_{i=1}^j \frac{1}{p_{\phi(i)}} = 1, & j = 1, \dots, n-1, \\ \frac{1}{r} + \sum_{i=1}^n \frac{1}{p_{\phi(i)}} = 1. \end{cases}$$

In the following estimates,  $\sum_{F_{\phi(1)}}$  runs over all  $F_{\phi(1)} \in \mathcal{F}_{\phi(1)}$  that satisfy (3.5) for fixed  $F_{\phi(i)} \in \mathcal{F}_{\phi(i)}$ ,  $i = 2, \dots, n$ .

$$\begin{aligned} & \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right) \sum_{\substack{Q: \\ \pi(Q) = (F_{\phi(i)})}} K(Q) \prod_{i=1}^n \sigma_{\phi(i)}(Q) \\ & \leq \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right) \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=1}^n \sigma_{\phi(i)}(Q) \\ & = \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right) \sigma_{\phi(1)}(F_{\phi(1)})^{1/p_{\phi(1)}} \\ & \quad \times \left( \int_{F_{\phi(1)}} \left( \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=2}^n \sigma_{\phi(i)}(Q) 1_Q \right) d\sigma_{\phi(1)} \right) \sigma_{\phi(1)}(F_{\phi(1)})^{1/r_1^\phi}, \end{aligned}$$

where we have used (3.6) with  $j = 1$ . By Hölder's inequality, we have further that

$$\begin{aligned} & \leq \left[ \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/p_{\phi(1)}} \\ & \quad \times \left[ \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} \left( \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=2}^n \sigma_{\phi(i)}(Q) 1_Q \right) d\sigma_{\phi(1)} \right)^{r_1^\phi} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/r_1^\phi}. \end{aligned}$$

By the same way as the estimate of  $(I_n)$ , we see that the last term is majorized by

$$C \left( \int_{F_{\phi(2)}} \left( \sum_{Q \subset F_{\phi(2)}} K(Q) \prod_{i=2}^n \sigma_{\phi(i)}(Q) 1_Q \right)^{r_1^\phi} d\sigma_{\phi(1)} \right)^{1/r_1^\phi}.$$

By Lemma 2.1, we have further that

$$\leq C \left( \sum_{Q \subset F_{\phi(2)}} K_1^\phi(Q) \prod_{i=2}^n \sigma_{\phi(i)}(Q) \right)^{1/r_1^\phi}.$$

By (2.3), we notice that

$$(3.7) \quad \frac{1}{r_i^\phi} + \frac{1}{p_{\phi(i)}} = \frac{1}{r_{i-1}^\phi}, \quad i = 2, \dots, n-1.$$

In the following estimates,  $\sum_{F_{\phi(2)}}$  runs over all  $F_{\phi(2)} \in \mathcal{F}_{\phi(2)}$  that satisfy, for fixed  $F_{\phi(i)} \in \mathcal{F}_{\phi(i)}$ ,  $i = 3, \dots, n$ ,

$$(3.8) \quad \pi_{\mathcal{F}_{\phi(j)}}(F_{\phi(i)}) = F_{\phi(j)} \quad \text{for all } 2 \leq i < j \leq n.$$

There holds

$$\begin{aligned}
& \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} f_{\phi(2)} d\sigma_{\phi(2)} \right) \times \left( \sum_{Q \subset F_{\phi(2)}} K_1^\phi(Q) \prod_{i=2}^n \sigma_{\phi(i)}(Q) \right)^{1/r_1^\phi} \\
& \quad \times \left( \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right)^{1/p_{\phi(1)}} \\
& = \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} f_{\phi(2)} d\sigma_{\phi(2)} \right) \sigma_{\phi(2)}(F_{\phi(2)})^{1/p_{\phi(2)}} \\
& \quad \times \left( \int_{F_{\phi(2)}} \left( \sum_{Q \subset F_{\phi(2)}} K_1^\phi(Q) \prod_{i=3}^n \sigma_{\phi(i)}(Q) 1_Q \right) d\sigma_{\phi(2)} \right)^{1/r_1^\phi} \sigma_{\phi(2)}(F_{\phi(2)})^{1/r_2^\phi} \\
& \quad \times \left( \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right)^{1/p_{\phi(1)}},
\end{aligned}$$

where we have used (3.7) with  $i = 2$ . Recall that (3.6) with  $j = 2$ . Then Hölder's inequality gives

$$\begin{aligned}
& \leq \left[ \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} f_{\phi(2)} d\sigma_{\phi(2)} \right)^{p_{\phi(2)}} \sigma_{\phi(2)}(F_{\phi(2)}) \right]^{1/p_{\phi(2)}} \\
& \quad \times \left[ \sum_{F_{\phi(2)}} \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/p_{\phi(1)}} \\
& \quad \times \left[ \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} \left( \sum_{Q \subset F_{\phi(2)}} K_1^\phi(Q) \prod_{i=3}^n \sigma_{\phi(i)}(Q) 1_Q \right) d\sigma_{\phi(2)} \right)^{r_2^\phi/r_1^\phi} \sigma_{\phi(2)}(F_{\phi(2)}) \right]^{1/r_2^\phi}.
\end{aligned}$$

The last term is majorized by

$$C \left( \int_{F_{\phi(3)}} \left( \sum_{Q \subset F_{\phi(3)}} K_2^\phi(Q) \prod_{i=3}^n \sigma_{\phi(i)}(Q) 1_Q \right)^{r_2^\phi/r_1^\phi} d\sigma_{\phi(2)} \right)^{1/r_2^\phi}.$$

By Lemma 2.1, we have further that

$$\leq C \left( \sum_{Q \subset F_{\phi(3)}} K_2^\phi(Q) \prod_{i=3}^n \sigma_{\phi(i)}(Q) \right)^{1/r_2^\phi}.$$

By being continued inductively until the  $n - 1$  step, we obtain

$$\begin{aligned}
(3.3) &\leq C \left[ \sum_{F_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{p_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/p_{\phi(n)}} \\
&\times \left[ \sum_{F_{\phi(n)}} \sum_{F_{\phi(n-1)}} \left( \int_{F_{\phi(n-1)}} f_{\phi(n-1)} d\sigma_{\phi(n-1)} \right)^{p_{\phi(n-1)}} \sigma_{\phi(n-1)}(F_{\phi(n-1)}) \right]^{1/p_{\phi(n-1)}} \\
&\times \vdots \\
&\times \left[ \sum_{F_{\phi(n)}} \sum_{F_{\phi(n-1)}} \cdots \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/p_{\phi(1)}} \\
&\times \left[ \sum_{F_{\phi(n)}} \left( \int_{F_{\phi(n)}} \left( \sum_{Q \subset F_{\phi(n)}} K_{n-1}^{\phi}(Q) 1_Q \right) d\sigma_{\phi(n)} \right)^{r/r_{n-1}^{\phi}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/r},
\end{aligned}$$

where  $\sum_{F_{\phi(n)}}$  runs over all  $F_{\phi(n)} \in \mathcal{F}_{\phi(n)}$  and  $\sum_{F_{\phi(k)}}$ ,  $k = 3, \dots, n-1$ , runs over all  $F_{\phi(k)} \in \mathcal{F}_{\phi(k)}$  that satisfy, for fixed  $F_{\phi(i)}$ ,  $i = k+1, \dots, n$ ,

$$(3.9) \quad \pi_{\mathcal{F}_{\phi(j)}}(F_{\phi(i)}) = F_{\phi(j)} \quad \text{for all } k \leq i < j \leq n.$$

The last term is majorized by

$$C \left( \int_{Q_0} \left( \sum_{Q \subset Q_0} K_{n-1}^{\phi}(Q) 1_Q \right)^{r/r_{n-1}^{\phi}} d\sigma_{\phi(n)} \right)^{1/r} \leq c_2.$$

It follows from (3.5), (3.8), (3.9) and the uniqueness of the parents that

$$\begin{aligned}
&\left[ \sum_{F_{\phi(n)}} \sum_{F_{\phi(n-1)}} \cdots \sum_{F_{\phi(i)}} \left( \int_{F_{\phi(i)}} f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F_{\phi(i)}) \right]^{1/p_{\phi(i)}} \\
&\leq \left[ \sum_{F_{\phi(n)}} \sum_{\substack{F_{\phi(i)} \in \mathcal{F}_{\phi(i)}: \\ \pi_{\mathcal{F}_{\phi(n)}}(F_{\phi(i)}) = F_{\phi(n)}}} \left( \int_{F_{\phi(i)}} f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F_{\phi(i)}) \right]^{1/p_{\phi(i)}} \\
&= \left[ \sum_{F_{\phi(i)} \in \mathcal{F}_{\phi(i)}} \left( \int_{F_{\phi(i)}} f_{\phi(i)} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F_{\phi(i)}) \right]^{1/p_{\phi(i)}} \\
&\leq C \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.
\end{aligned}$$

Altogether, we obtain

$$(3.3) \leq C c_2 \prod_{i=1}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.$$

This yields (a) of Theorem 1.4.

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